

ON SUBELLIPTIC MANIFOLDS

SHULIM KALIMAN, FRANK KUTZSCHEBAUCH AND TUYEN TRUNG TRUONG

*Dedicated to Mikhail Zaidenberg
on the occasion of his 70-th birthday*

ABSTRACT. A smooth complex quasi-affine algebraic variety Y is flexible if its special group $\mathrm{SAut}(Y)$ of automorphisms (generated by the elements of one-dimensional unipotent subgroups of $\mathrm{Aut}(Y)$) acts transitively on Y . An irreducible algebraic manifold X is locally stably flexible if it is the union $\bigcup X_i$ of a finite number of Zariski open sets, each X_i being quasi-affine, so that there is a positive integer N for which $X_i \times \mathbb{C}^N$ is flexible for every i . The main result of this paper is that the blowup of a locally stably flexible manifold at a smooth algebraic submanifold (not necessarily equi-dimensional or connected) is subelliptic, and hence Oka. This result is proven as a corollary of some general results concerning the so-called k -flexible manifolds.

CONTENTS

Introduction	1
1. Sprays and subellipticity	3
2. Flexible manifolds	5
2.1. Proof of Theorem 2.4.	9
3. First facts about sprays on flexible manifolds.	10
4. Affine modifications	12
5. k -Flexibility	15
6. Main theorems	17
References	18

INTRODUCTION

The notion of a subelliptic manifold (i.e. a manifold which admits a dominating family of sprays) was introduced by Forstnerič in [6], inspired by hints from Gromov in [9]. It is a natural generalization to the stronger condition of admitting a single dominating spray, called elliptic. The importance of the notion of subellipticity is that as in the case of elliptic manifolds it implies all Oka properties. In other words such a

Date: January 31, 2017.

The second author was partially supported by Schweizerische Nationalfonds grants No. 200020-134876/1 and 200021-140235/1 and the third author was supported by Australian Research Council grants DP120104110 and DP150103442.

2010 *Mathematics Subject Classification:* 14R20, 32M17.

Key words: affine varieties, Oka principle, algebraic subellipticity, Oka manifolds, group actions, one-parameter subgroups, transitivity.

subelliptic manifold is X is an Oka manifold as proven by Forstnerič in [6]. In particular being an Oka manifold implies that every holomorphic map from a convex domain K in \mathbb{C}^n into X can be approximated (in the compact-open topology) by a holomorphic map from \mathbb{C}^n to X . Needless to say that this leads to many remarkable consequences (e.g., see [5]). On the other hand, having the same consequences, subellipticity is easier to establish than ellipticity, which is exemplified by the main results of the present paper.

The simplest example of an elliptic manifold is, of course, the Euclidean space \mathbb{C}^n itself. Furthermore, Gromov proved ellipticity in the case of the complement to a subvariety of codimension at least 2 in \mathbb{C}^n . Any algebraic manifold which is locally isomorphic to such complements (resp. \mathbb{C}^n) is called a manifold of class \mathcal{A} (resp. \mathcal{A}_0) ([5, Definition 6.4.5.], [16, Remark 3]). Since in the algebraic case subellipticity turns out to be a local property we see that a manifold of class \mathcal{A} is always subelliptic. Gromov observed also the following.

Proposition 0.1. *Let X be a complex manifold of class \mathcal{A}_0 and Y be the result of blowing X up at a finite number of points. Then Y is also a manifold of class \mathcal{A}_0 and, therefore, subelliptic.*

For example this yields subellipticity of compact rational surfaces (see [5, Corollary 6.4.8]).

There were no analogs of Proposition 0.1 until the recent paper of Lárusson and the third author [16] who proved the following.

Theorem 0.2. *Let X be an algebraic manifold of class \mathcal{A} and $\pi : \tilde{X} \rightarrow X$ be the blowing up of X along a smooth algebraic (not necessarily connected) submanifold of codimension at least 2. Then \tilde{X} is subelliptic.*

The proofs in [16] made use crucially of the fact that \mathbb{C}^n has a lot of automorphisms, as first pointed out by Gromov and generalised by Winkelmann. This last property is shared by the so-called flexible manifolds, extensively-studied in affine algebraic geometry. Recall that one of equivalent definitions states that a smooth complex quasi-affine algebraic variety X of dimension at least 2 is flexible if its special group $\text{SAut}(X)$ of automorphisms (generated by the elements of one-dimensional unipotent subgroups of $\text{Aut}(X)$) acts transitively on X . It is easy to establish that flexible manifolds are algebraically subelliptic (and even algebraically elliptic). Furthermore, there is no need to discuss complements to subvarieties of codimension at least 2 in flexible manifolds because such complements are again flexible ([4]). This observation was a strong indication for us that the above construction can survive replacement of Euclidean spaces by flexible manifolds. This is in fact true, and we can actually prove the same result for a more general class of manifolds which we are going to define next.

Definition. An irreducible algebraic manifold X is locally stably flexible, if it is the union $\bigcup X_i$ of a finite number of Zariski open sets, each X_i being quasi-affine, so that there is a positive integer N for which $X_i \times \mathbb{C}^N$ is flexible for every i .

Our main result is the following.

Theorem 0.3. *Let X be a locally stably flexible manifold. Suppose that $\pi : \tilde{X} \rightarrow X$ is the blowing up of X along a smooth algebraic submanifold Z , not necessarily equi-dimensional or connected. Then \tilde{X} is algebraically subelliptic.*

After the Theorems 0.2 and 0.3, it is natural to ask the following questions concerning the behavior under blowups of various classes of algebraic manifolds of interest in affine geometry and Oka theory.

Q1. Is the class of algebraically subelliptic manifolds preserved by blowups? We note that by Gromov's results the class \mathcal{A} (which is a smaller class of manifolds) and by results in [16] the class of strongly algebraically dominated manifolds (which is a bigger class of manifolds) are both preserved by blowups.

Q2. Is the class of locally stably algebraic manifolds preserved by blowups?

The remaining of this paper is organized as follows. In Section 1 we remind the notions of sprays and subellipticity and establish some simple facts which are immediate consequences of the results presented in [5]. In Section 2 we describe technique developed for flexible manifolds in [4] and [1] and prove a non-trivial fact (Theorem 2.4) heavily based on [4]. This theorem deals with a partial quotient morphism $\varrho : X \rightarrow Q$ of a flexible manifold X with respect to some G_a -action. It establishes that up to an automorphism of X for every closed submanifold Z of X (of codimension at least 2) and every $z \in Z$ we can suppose that $\varrho|_Z : Z \rightarrow \varrho(Z)$ is a local isomorphism over a Zariski neighborhood of $\varrho(z) \in \varrho(Z)$. In Section 3 we prove simple facts which, in particular, include ellipticity of flexible manifolds. Section 4 is devoted to technique of affine modifications which can be mostly found in [13]. It is necessary because in the above notations X is an affine modification of $Q \times \mathbb{C}$. It turns out that we need to present X as a more refined affine modification for which we introduce in Section 5 the notion of k -flexibility. Namely, X is k -flexible if for some dominant morphism $\tau : X \rightarrow P$ there is a Zariski dense open subset P_0 of P for which $\tau^{-1}(P_0)$ is isomorphic to $P_0 \times \mathbb{C}^k$ which implies that X is an affine modification of $P \times \mathbb{C}^k$.¹ With all preparations done we obtain our main theorems in Section 6.

Acknowledgments. The authors would like to thank Finnur Lárusson for his helpful comment concerning the descent property of algebraic subellipticity. Most of this work was done during a stay of the second author at the University of Miami and he thanks this institution for its hospitality and excellent working conditions. It was also partially done while the second and third authors were attending the program "Workshop on higher algebraic geometry, holomorphic dynamics and their interaction" at the Institute for Mathematical Sciences, National University of Singapore in January 2017, and we thank the institution and the organizers for their hospitality and financial support.

1. SPRAYS AND SUBELLIPTICITY

Let us remind some definitions which can be found in [5].

¹It is worth mentioning that most of flexible manifolds are k -flexible for some $k \geq 2$. Actually, starting from dimension 3 we do not know examples of flexible manifolds that are not at least 2-flexible.

Definition 1.1. (i) A holomorphic vector bundle $p : E \rightarrow X$ over a complex manifold X is called a spray if there exists a holomorphic map $s : E \rightarrow X$ such that for every point y in the zero section S of E one has $s(y) = p(y) := x$. That is, a spray is a triple (E, p, s) .

(ii) A spray is called dominating if for every $y \in S$ one has $ds(T_y p^{-1}(x)) = T_x X$.

(iii) A family of sprays $\{E_i, p_i, s_i\}_{i=1}^m$ on X is called dominating if for every $y \in S$

$$ds_1(T_y p_1^{-1}(x)) + ds_2(T_y p_2^{-1}(x)) + \dots + ds_m(T_y p_m^{-1}(x)) = T_x X.$$

(iv) A complex manifold X is called elliptic (resp. subelliptic) if it admits a dominating holomorphic spray (resp. a dominating family of holomorphic sprays).

(v) We say that a spray (E, p, s) is of rank k if the rank of the vector bundle $p : E \rightarrow X$ is k .

Convention 1.2. From now on we consider only **algebraic** sprays (E, p, s) on algebraic complex manifolds which means that the vector bundle $p : E \rightarrow X$ is algebraic and the map $s : E \rightarrow X$ is algebraic. We omit this adjective “algebraic” below.

Under this convention the following definition makes sense.

Definition 1.3. (a) Let X_0 be a nonempty Zariski open subset of a complex algebraic manifold X . An algebraic vector bundle $p : E \rightarrow X_0$ is called a spray on X_0 with values in X if there exists a holomorphic map $s : E \rightarrow X$ such that for every point $x \in X_0$ in the zero section S_0 of E one has $s(x) = x$.

(b) Let $s' : E \rightarrow X_0$ be another spray on X_0 with values in X where $p : E \rightarrow X_0$ is the same vector bundle as in (a). We say that it is equivalent to the spray s from (a) if for general points $y \in S_0$ and $x = p(y)$ there is a linear automorphism λ of the fiber $E_x = p^{-1}(x)$ for which

$$(1) \quad s \circ \lambda|_{E_x} = s'|_{E_x}$$

(c) The notion of a dominating spray on X_0 with values in X is described exactly as in Definition 1.1 with y running over a section S_0 of $p : E \rightarrow X_0$. In the same fashion we deal with a dominating family of sprays on X_0 with values in X .

Convention 1.2 enables us to use the following facts (see, the proof of [5, Theorem 6.4.2]) again and again.

Proposition 1.4. *Let $s : E \rightarrow X$ be a spray on X_0 with values in X as in Definition 1.3. Then there exists an equivalent spray $s' : E \rightarrow X$ on X_0 with values in X such that it extends to a spray on X . Furthermore, if $X \setminus X_0$ is a principal divisor then this spray s' can be chosen so that equation (1) holds for every $x \in X_0$.*

Corollary 1.5. *Let $\{U_i\}_i$ be a cover of a complex algebraic manifold X by Zariski open sets such that for every i there is a dominating family of sprays on U_i with values in X . Then there is a dominating family of sprays on X .*

Corollary 1.6. *Let $s : E \rightarrow X$ be a spray on X as in Definition 1.1 and let $\varphi : X \rightarrow Y$ be a birational morphism which yields an isomorphism between Zariski open subsets*

$X_0 \subset X$ and $Y_0 \subset Y$, i.e. one has the following commutative diagram

$$\begin{array}{ccc} E|_{X_0} & \xrightarrow{\psi} & F \\ \downarrow p|_{X_0} & & \downarrow q \\ X_0 & \xrightarrow{\varphi|_{X_0}} & Y_0. \end{array}$$

of isomorphic vector bundles. Then $r = \varphi \circ s \circ \psi^{-1} : F \rightarrow Y$ is a spray on Y_0 with values in Y . Furthermore, if the complement of Y_0 in Y is a principal divisor there is an equivalent spray $r' : F \rightarrow Y$ extendable to a spray on Y and such that $r(q^{-1}(y)) = r'(q^{-1}(y))$ for every $y \in Y_0$.

Corollary 1.7. *The class of manifolds admitting sprays is closed with respect to the procedure of blowing down.*

2. FLEXIBLE MANIFOLDS

Recall the following facts which can be found in [1].

Definition 2.1. (1) A derivation σ on the ring A of regular functions on a quasi-affine algebraic manifold X is called locally nilpotent if for every $0 \neq a \in A$ there exists a natural n for which $\sigma^n(a) = 0$. For the smallest n with this property one defines the degree of a with respect to σ as $\deg_\sigma a = n - 1$. This derivation can be viewed as a vector field on X which we also call locally nilpotent. The phase flow of this vector field is an algebraic G_a -action on X , i.e. the action of the group \mathbb{C}_+ of complex numbers with respect to addition which can be viewed as a one-parameter unipotent group U in the group $\text{Aut}(X)$ of all algebraic automorphisms of X . In fact, every G_a -action is generated by a locally nilpotent vector field (e.g, see [7]).

(2) A quasi-affine manifold X is called flexible if for every $x \in X$ the tangent space $T_x X$ is spanned by the tangent vectors to the orbits of one-parameter unipotent subgroups of $\text{Aut}(X)$ through x .

(3) The subgroup $\text{SAut}(X)$ of $\text{Aut } X$ generated by all one-parameter unipotent subgroups is called special.

We have the following [1], [4].

Proposition 2.2. *For every irreducible quasi-affine algebraic variety X the following are equivalent*

- (i) *the special subgroup $\text{SAut}(X)$ acts transitively on X_{reg} ;*
- (ii) *the special subgroup $\text{SAut}(X)$ acts infinitely transitively on X_{reg} (i.e. for every natural m the action is m -transitive);*
- (iii) *X_{reg} is flexible.*

By the Rosenlicht Theorem (see [18, Theorem 2.3]) for X , A , and U as in Definition 2.1 one can find a finite set of U -invariant functions $a_1, \dots, a_m \in A$, which separate general U -orbits in X . They generate a morphism $\varrho : X \rightarrow Q$ into an affine algebraic variety Q . Note that this set of invariant functions can be chosen so that Q is normal (since X is normal).

Definition 2.3. Such a morphism $\varrho : X \rightarrow Q$ into a normal Q will be called a partial quotient. In the case when a_1, \dots, a_m generate the subring A^U of U invariant elements of A such a morphism is called the categorical quotient.²

The main aim of this section is the next theorem.

Theorem 2.4. *Let Z be a submanifold of codimension at least 2 in a flexible affine algebraic manifold X , and σ be a nontrivial locally nilpotent vector field on X . Suppose that $\varrho : X \rightarrow Q$ is a partial quotient morphism of the G_a -action associated with σ such that Q is a normal variety. Then for every finite set $z_1, \dots, z_m \in Z$ of distinct points one can find an automorphism α of X such that for every $i = 1, \dots, m$ and $\varrho_\alpha = \varrho \circ \alpha$ one has*

- (i) *the point $\varrho_\alpha(z_i)$ is general (and, therefore, smooth) in Q ,*
- (ii) *$\varrho_\alpha(z_i)$ is contained in a smooth part of $\varrho_\alpha(Z)$;*
- (iii) *the morphism $\varrho_\alpha|_Z : Z \rightarrow \varrho_\alpha(Z)$ is local embedding at z_i (and, in particular, $\varrho_\alpha|_Z$ is birational).*

Furthermore, let $\tau : Q \rightarrow P$ be a morphism such that $\dim P = \dim Z$. Then α can be chosen so that

- (iv) *$\tau|_{\varrho_\alpha(Z)} : \varrho_\alpha(Z) \rightarrow P$ is étale at $\varrho_\alpha(z_i)$ for every i .*

The proof of this fact is heavily based on the technique from [4] and it requires some preparations, but first let us extract some corollary using following notion introduced by Ramanujam [19].

Definition 2.5. Given irreducible algebraic varieties X and A and a map $\varphi : A \rightarrow \text{Aut}(X)$ we say that (A, φ) is an *algebraic family of automorphisms on X* if the induced map $A \times X \rightarrow X$, $(\alpha, x) \mapsto \varphi(\alpha).x$, is a morphism.

Remark 2.6. Note that properties (i) -(iv) from Theorem 5.6 survive under a small perturbation of the automorphism α in an irreducible algebraic family of automorphisms A . This implies that they are valid for a Zariski open subset of A (because of the algebraicity) and we have the following.

Corollary 2.7. *Let Theorem 2.4 hold for an automorphism α which is contained in an irreducible algebraic family of automorphisms A . Then the statement remains valid if one replaces α by a general element of A .*

The first step in the proof of Theorem 2.4 is the following.

Lemma 2.8. *Let the assumption of Theorem 2.4 hold. Then for every finite set $z_1, \dots, z_m \in Z$ of distinct points one can find an automorphism α of X such that for every $i = 1, \dots, m$ and $\varrho_\alpha = \varrho \circ \alpha$ properties (i) and (iv) from Theorem 2.4 are true. Furthermore,*

- (ii') *$\varrho|_{Z_\alpha} : Z_\alpha \rightarrow \varrho(Z_\alpha)$ is a local embedding at every point $\alpha(z_i)$.*

²However, in general A^U is not finitely generated by the Nagata's example. That is, why, following [4] we prefer to work with partial quotients.

Proof. Choose general points q_1, \dots, q_m in Q and general points x_1, \dots, x_m in X for which $\varrho(x_i) = q_i$. Let $(v_{i,1}, \dots, v_{i,n})$ (resp. $(u_{i,1}, \dots, u_{i,n+1})$) be a local analytic coordinate system at $q_i \in Q$ (resp. $x_i \in X$) such that $\varrho^*(v_{i,j}) = u_{i,j}$. By [1, Theorem 4.14 and Remark 4.16] one can choose an automorphism α of X such that $\alpha(z_i) = x_i$, $i = 1, \dots, m$ and, furthermore, $\alpha(Z)$ is tangent to the subvariety $u_{i,n-k+1} = \dots = u_{i,n+1} = 0$ where $k+1$ is the codimension of Z in X . By construction α satisfies properties (i) and (ii').

For (iv) it suffices to require that $(v_{i,1}, \dots, v_{i,n-k})|_{\varrho_\alpha(Z)}$ is a lift of a local analytic coordinate system on P under τ and we are done. \square

Definition 2.9. For every locally nilpotent vector fields σ and each function $f \in \text{Ker } \sigma$ from its kernel the field $f\sigma$ is called a replica of σ . Recall that such a replica is automatically locally nilpotent.

Proposition 2.10. (cf. [4]) *Let δ_0 be a locally nilpotent vector field on a quasi-affine algebraic manifold X , $\varrho_0 : X \rightarrow Q_0$ be an associated partial quotient morphism, x be a general point of X , and O_1 be the orbit of x under the phase flow of δ_0 . Then there exists a locally nilpotent vector field δ_1 such that*

(#) *for general points $x_1, \dots, x_{n-1} \in O_1$ and the vectors $\delta_{1,x_1}, \dots, \delta_{1,x_{n-1}}$ (which are the values of δ_1 at these points) the vectors v_1, \dots, v_{n-1} form a basis of $T_{q_0}Q_0$ where $q_0 = \varrho_0(x)$ and $v_i = d\varrho_0(\delta_{1,x_i})$.*

Furthermore, let condition (#) hold and H be the group of algebraic automorphisms of X generated by the elements from the phase flows of δ_0 , δ_1 , and their replicas. Then the orbit of x under the action of H is Zariski open in X .

Proof. By [1, Theorem 4.14 and Remark 4.16] there exists an automorphism $\alpha \in \text{Aut}(X)$ such that it fixes points x_1, \dots, x_{n-1} and for every i the linear map $d\alpha|_{T_{x_i}X}$ coincides with a prescribed element of \mathbf{SL}_{n-1} . Furthermore, for every fixed $k \in \mathbb{N}$ (however large it is) we can require that the k -jet $\alpha_{x_i}^k$ of α at x_i coincides with such a linear part. Hence choosing any locally nilpotent derivation δ_1 for which every $\delta_{1,x_i} \neq 0$ we can achieve (#) replacing δ_1 by $\alpha_*(\delta_1)$.

For every nonzero locally nilpotent δ_0 the statement of [4, Proposition 1.14] yields the existence of δ_1 for which the orbit of x under the action of the group H is open. However, the analysis of the proof of this fact shows that any δ_1 satisfying (#) fits this purpose. \square

Remark 2.11. (a) In fact, we can replace condition (#) in Proposition 2.10 with the following. For every point $y \in O_1$ denote by $y(t)$ the image of y under the action of the phase flow of δ_0 at time t . Then we can require that

(#') *for a general moment of time t and general $y \in O_1$ the vector $d\varrho_0(\delta_{1,y(t)}) - d\varrho_0(\delta_{1,y(t)})$ is general in $T_{q_0}Q_0$.*

The fact that (#') implies (#) is clear. Indeed, otherwise there is a proper subspace $V \subset T_{q_0}Q_0$ such that for every point $w \in O_1$ one has $d\varrho_0(\delta_{1,w}) \in V$. Then $d\varrho_0(\delta_{1,y(t)}) -$

$d\varrho_0(\delta_{1,y(t)})$ must be also in V which implies that this vector cannot be general. A contradiction.

To assure $(\#')$ one can choose x_1, \dots, x_{n-1} so that $x_i = x(it_0)$ for some nonzero t_0 and choose an automorphism α so that after its application the vectors $d\varrho_0(\delta_{1,x_i}) - d\varrho_0(\delta_{1,x_{i-1}})$, $i = 1, \dots, n-1$ form a basis of $T_{q_0}Q_0$ (where $x_0 = x$).

(b) Moreover, the proof of Proposition 2.10 implies that one can choose δ_1 so that condition $(\#')$ holds not only for the orbit O_1 of one general point $x \in X$ but simultaneously for the orbits of any finite set of general points in X .

We need some further facts from [4].

Notation 2.12. (a) Denote by U^i the unipotent one-parameter subgroup associated with δ_i from Proposition 2.10 and for every $f \in \text{Ker } \delta_0 \setminus \text{Ker } \delta_1$ (resp. $g \in \text{Ker } \delta_1 \setminus \text{Ker } \delta_0$) denote by U_f^0 (resp. U_g^1) the one-parameter group associated with the replica $f\delta_0$ (resp. $g\delta_1$).

(b) To any sequence of invariant functions

(2)

$$\mathcal{F} = \{f_1, \dots, f_s, g_1, \dots, g_s\}, \text{ where } f_i \in \text{Ker } \delta_1 \setminus \text{Ker } \delta_0 \text{ and } g_i \in \text{Ker } \delta_0 \setminus \text{Ker } \delta_1,$$

we associate an algebraic family of automorphisms $\mathbb{C}^{2s} \rightarrow \text{Aut}(X)$ defined by the product

$$(3) \quad U^{\mathcal{F}} = U_{f_s}^1 \cdot U_{g_s}^0 \cdot \dots \cdot U_{f_1}^1 \cdot U_{g_1}^0 \subseteq H.$$

More generally, given a tuple $\kappa = (k_i, l_i)_{i=1, \dots, s} \in \mathbb{N}^{2s}$ the product

$$(4) \quad U_{\kappa} = U_{\kappa}^{\mathcal{F}} = U_{f_s}^{k_s} \cdot U_{g_s}^{l_s} \cdot \dots \cdot U_{f_1}^{k_1} \cdot U_{g_1}^{l_1} \subseteq H$$

yields as well an algebraic family of automorphisms.

Proposition 2.13. ([4, Corollary 4.4]) *There is a finite collection of invariant functions \mathcal{F} as in (2) such that for any sequence $\kappa = (k_i, l_i)_{i=1, \dots, s} \in \mathbb{N}^{2s}$ the algebraic family of automorphisms U_{κ} as in (4) has a dense open orbit in X . This orbit $O(U_{\kappa})$ coincides with $O(H)$ and so does not depend on the choice of $\kappa \in \mathbb{N}^{2s}$.*

Remark 2.14. Let $\tilde{\mathcal{F}} = \{\tilde{f}_1, \dots, \tilde{f}_r, f_1, \dots, f_s, \tilde{g}_1, \dots, \tilde{g}_r, g_1, \dots, g_s\}$ where f_i and g_i are as in Notation 2.12 while $\tilde{f}_i \in \text{Ker } \delta_1 \setminus \text{Ker } \delta_0$ and $\tilde{g}_i \in \text{Ker } \delta_0 \setminus \text{Ker } \delta_1$. Note that if $O(U_{\kappa}^{\mathcal{F}}) = O(H)$ as in Proposition 2.13 then one has also $O(U_{\kappa}^{\tilde{\mathcal{F}}}) = O(H)$.

Notation 2.15. (a) Given a one-parameter group $U \in \mathcal{U}(X)$ we let $U^* = U \setminus \{\text{id}\}$. Given a collection $\mathcal{F} = \{f_1, \dots, f_s, g_1, \dots, g_s\}$ of invariant functions as in Notation 2.12 and $U_{\kappa} = U_{f_s}^{k_s} \cdot U_{g_s}^{l_s} \cdot \dots \cdot U_{f_1}^{k_1} \cdot U_{g_1}^{l_1}$ as in (4), we let

$$U_{\kappa}^* = U_{f_s}^{1*} \cdot U_{g_s}^{0*} \cdot \dots \cdot U_{f_1}^{1*} \cdot U_{g_1}^{0*}.$$

(b) Consider δ_0 as in Notation 2.12 and its partial quotient morphism $\varrho_0 : X \rightarrow Q_0$. Then it can be extended to a proper morphism $\bar{\varrho}_0 : \bar{X} \rightarrow Q_0$. There is exactly one (so-called horizontal) irreducible component D_0 of the variety $\bar{X} \setminus X$ for which the restriction of the morphism $\bar{\varrho}|_{D_0} : D_0 \rightarrow Q$ is birational [4].

Proposition 2.16. *Let the assumption of Proposition 2.13 hold and Z be a closed subvariety of X of codimension at least 2. Then the integers $k_1, l_1, k_2, l_2, \dots, k_s, l_s$ in Notation 2.15 can be chosen so that there exists a proper subvariety $R \subset D_0$ such that for every element $\alpha \in U_\kappa^*$ the closure \bar{Z}_α of $\alpha(Z)$ in \bar{X} meets D_0 from Notation 2.15 along R only.*

Proof. The statement is a special case of [4, Proposition 4.11]). \square

Lemma 2.17. *Let the assumption of Theorem 2.4 hold and let $U_\kappa, \varrho_0, \bar{\varrho}_0, D_0$ be as Proposition 2.16 with $\delta_0 = \sigma$. Then there exists an automorphism $\alpha \in \text{Aut}(X)$ such that*

(a) *the conclusions of Lemma 2.8 are satisfied for every point $z'_i = \alpha(z_i)$ in $Z' = \alpha(Z)$ and $\varrho_0(z'_i) \notin \bar{\varrho}_0(\bar{Z}' \cap D_0) \subset \bar{\varrho}_0(R')$ where R' plays the same role for Z' as R for Z in Proposition 2.16.*

Furthermore, let $O_i = \varrho_0^{-1}(\varrho_0(z'_i))$ $i = 1, \dots, m$, $M_i \subset O_i$ be a finite subset, and B_y be the analytic branch of Z' at $y \in M_i$. Then for every i and every such a point y

(b) *the vector $d\varrho_0(\delta_{1,z'_i}) - d\varrho_0(\delta_{1,y})$ is not contained in the tangent cone of $\varrho_0(B_y)$ at $\varrho_0(z'_i)$.*

Proof. Choose an automorphism $\beta \in \text{Aut}(X)$ such that the conclusions of Lemma 2.8 hold for every point $z''_i = \beta(z_i)$ of the variety $Z'' = \beta(Z)$. That is, each z''_i is a general point of X , $\varrho|_{Z''} : Z'' \rightarrow \varrho(Z'')$ is a local embedding at z''_i , and $\varrho(z''_i)$ is contained in a smooth analytic branch of $\varrho(Z'')$ which is in turn contained in the smooth part of Q . Furthermore, perturbing α by the virtue of [1, Proposition 4.14 and Remark 4.16] we can suppose that condition (b) holds with Z' and z'_i replaced by Z'' and z''_i . Note that this conditions are preserved by perturbation of Z'' by any automorphism $\gamma \in U_\kappa^*$ sufficiently close to the identical map (in the compact-open topology). That is the conclusions of Lemma 2.8 are still valid for $Z' = \gamma(Z'')$ and $z'_i = \gamma(z''_i)$. Note also that by Proposition 2.16 $\varrho_0(z'_i) \notin \bar{\varrho}_0(R')$ for any general γ . Thus one has now (a) since the orbit of any general point z''_i under the action of U_κ^* is open by Proposition 2.13. Since condition (b) is preserved for general γ we are done. \square

2.1. Proof of Theorem 2.4. Treat $\sigma, z_1 \in Z$, and $\varrho^{-1}(\varrho(z_1))$ as $\delta_0, z'_1 \in Z'$ and O_1 in Lemma 2.17. Choose δ_1 so that the assumptions of Proposition 2.10 and Remark 2.11 are satisfied for a general point $y \in O_1 \simeq \mathbb{C}$, and, in particular, the vector $d\varrho_0(\delta_{1,x(t)}) - d\varrho_0(\delta_{1,y(t)})$ is general in $T_{q_0}Q_0$ for general moment of time t . Suppose that a finite set $M \subset O_1$ consists of all non-general points.

Choose U_κ, R' , and Z' as in the proof of Lemma 2.17 (in particular condition (b) is valid for points in M). Keep in mind that for general $\alpha \in U_\kappa$ the conclusions of Lemma 2.8 are satisfied. That is, z'_1 is a general point of X , $\varrho_0|_{Z'} : Z' \rightarrow \varrho_0(Z')$ is a local embedding at z'_1 , and $\varrho_0(z'_1)$ is contained in a smooth analytic branch of $\varrho_0(Z')$ which is in turn contained in the smooth part of Q . For Theorem 2.4 we only need to establish that $\varrho_0|_{Z'}$ is a birational morphism on its image and $\varrho_0(Z')$ is unibranch at $q_1 = \varrho_0(z'_1)$ which is now equivalent to the fact that for the set $\varrho_0^{-1}(q_1) \cap Z'$ contains no other points but z'_1 .

We are going to show that this equivalent fact is true under replacement of Z' by $\alpha(Z')$ for a general $\alpha \in U_\kappa$. Suppose that $f_1 = g_1 = 1$ for $f_1, g_1 \in \mathcal{F}$ from Notation 2.12 which is allowed by Remark 2.14. Let us look for this α in the form $\alpha = \beta\alpha_1\alpha_0$ where $\alpha_0 \in U_{g_1^{l_1}}^{0*}$ and $\alpha_1 \in U_{f_1^{k_1}}^{1*}$. Furthermore, we suppose that β is as close to the identical automorphism as we wish and therefore $\alpha(Z')$ is as close to $\alpha_1\alpha_0(Z')$ near point z'_1 as we wish.

Assume that exists a point $y_1 \in \varrho_0^{-1}(q_1) \cap (Z' \setminus z'_1)$ (since otherwise we are done). Let B be the analytic branch of Z' at y_1 and $B_0 = \varrho_0(B)$ be the analytic branch of $\varrho_0(Z')$ at q_1 . Note that for any element α_0 we have equality $B_0 = \varrho_0(\alpha_0(B))$.

Case 1. Suppose that y_1 does not belong to M . By Remark 2.11 after application of a general α_0 the vector

$$\nu = d\varrho_0(d\alpha_1(\delta_{1,y_1})) - d\varrho_0(d\alpha_1(\delta_{1,z'_1}))$$

is general. That is, ν is not contained in the tangent cone $C_{q_1}B_0$ to B_0 at q_1 .

Denote by B'_0 the image of $\alpha(B)$ under ϱ_0 . Observe that if α_1 is close to the identity isomorphism then up to infinitesimals $\varrho_0(\alpha_1\alpha_0(z'_1))$ changes from q_1 in direction vector $cd\varrho_0(d\alpha_1(\delta_{1,z'_1}))$ (where c is a nonzero coefficient) while the cone $C_{\varrho_0(\alpha_1(y_1))}B'_0$ is obtained from the cone $C_{q_1}B_0$ via the shift by vector $cd\varrho_0(d\alpha_0(\delta_{1,y_1}))$.

This implies that for a general α_1 the point $\varrho_0(\alpha_1 \circ \alpha_0(z'_1))$ is not contained in $\varrho_0(\alpha_1 \circ \alpha_0(B))$, i.e. $\alpha_1 \circ \alpha_0(B)$ does not meet $\varrho_0^{-1}(\varrho_0(\alpha_1 \circ \alpha_0(z'_1)))$. Since β is close to the identical map as we wish the same is true for general $\alpha = \beta \circ \alpha_1 \circ \alpha_0$, i.e. $\alpha(B)$ does not meet $\varrho_0^{-1}(\varrho_0(\alpha(z'_1)))$. Therefore, $\alpha(Z')$ does not meet $\varrho_0^{-1}(\varrho_0(\alpha(z'_1)))$ near any point $y_1 \in \varrho_0^{-1}(q_1) \cap (Z \setminus z_1)$ for general α .

Case 2. Suppose that $y_1 \in M$ is not general in $\varrho_0^{-1}(q_1)$. Then we can suppose that ν is not contained in the tangent cone $C_{q_1}B_0$ to B_0 at q_1 by Lemma 2.17 (b) which yields the same conclusion that $\alpha(Z')$ does not meet $\varrho_0^{-1}(\varrho_0(\alpha(z'_1)))$ near any point $y_1 \in \varrho_0^{-1}(q_1) \cap (Z \setminus z_1)$ for general α .

Note also that since $\varrho_0(z'_1) \notin \bar{\varrho}_0(\overline{Z'} \cap D_0) \subset \bar{\varrho}_0(R')$ the points in $\varrho_0^{-1}(\varrho_0(\alpha(z'_1))) \cap \alpha(Z')$ depend upper-semicontinuously on α . Hence the previous argument shows that $\varrho_0^{-1}(\varrho_0(\alpha(z'_1))) \cap \alpha(Z' \setminus z'_1) = \emptyset$, i.e. we have the statements (i)-(iv) of Theorem 2.4 for z_1 and general α .

Note that in this argument we used only the fact that the restriction of δ_1 to the orbit O_1 of z_1 under U^0 satisfies condition $(\#')$ from Remark 2.11. However, the same Remark 2.11 shows that δ_1 can be chosen so that it satisfies $(\#')$ for the orbit of every z_i , $i = 1, \dots, m$. Thus, for a general α we have the statements (i)-(iv) of Theorem 2.4 for each z_i which concludes the proof. \square

3. FIRST FACTS ABOUT SPRAYS ON FLEXIBLE MANIFOLDS.

Theorem 3.1. *Every flexible manifold X is elliptic.*

Proof. Recall that for every $x \in X$ and each nonzero vector $v \in T_x X$ there is a locally nilpotent vector field σ on X for which the value σ_x of σ at x coincides with v [1, Corollary 4.3]. In particular, we find locally nilpotent vector fields $\sigma_1, \dots, \sigma_n$ for which

$\sigma_{1,x}, \dots, \sigma_{n,x}$ is a basis in $T_x X$. This implies that there is an open Zariski dense subset U of X so that for every point $y \in U$ the vectors $\sigma_{1,y}, \dots, \sigma_{n,y}$ is a basis in $T_y X$. Note that $\dim(X \setminus U) = m \leq n - 1$. Finding locally nilpotent vector fields that form a basis at general points of each of the component of $X \setminus U$ we can extend our sequence of vector fields to $\sigma_1, \dots, \sigma_n, \sigma_{n+1}, \dots, \sigma_l$ such that there is a Zariski open set $V \supset U$ for which $\dim(X \setminus V) < m$ and such that for every $y \in V$ these fields generate $T_y X$. Thus using induction by dimension we can suppose that $\sigma_1, \dots, \sigma_n, \sigma_{n+1}, \dots, \sigma_l$ generate $T_y X$ at every $y \in X$.

Let U^i be the one-parameter group of algebraic automorphism associated with σ_i and let U_t^i be the element of this group for the value of the time parameter t . Consider the trivial vector bundle $\pi : E \rightarrow X$ of rank l , i.e. for every $x \in X$ the fiber $E_x = \pi^{-1}(x)$ is isomorphic to \mathbb{C}^l with coordinates (t_1, \dots, t_l) .

Define the morphism $s : E \rightarrow X$ by the formula

$$(x, t_1, \dots, t_n) \rightarrow U_{t_1}^1 \circ \dots \circ U_{t_l}^l(x).$$

By construction, s is a dominating spray and we are done. □

By Corollary 1.5 we have the following.

Corollary 3.2. *Every locally flexible algebraic manifold is subelliptic.*

Notation 3.3. Let $\pi : \tilde{X} \rightarrow X$ be the blowing up of an affine manifold X along a closed smooth algebraic submanifold Z of codimension $k + 1 \geq 2$. Suppose that E is the exceptional divisor of π , i.e. $\pi|_E : E \rightarrow Z$ is a locally trivial fibration with fiber \mathbb{P}^k . Note that $\mathbb{C}[\tilde{X}] = \mathbb{C}[X]$ and \tilde{X} is a semi-affine manifold. In particular, the notion of locally nilpotent derivation (= vector field) on \tilde{X} is well-defined.

Proposition 3.4. *For every $z \in Z$, $x \in \pi^{-1}(z)$, and a nonzero vector $w \in T_x \tilde{X}$ tangent to $\pi^{-1}(z)$ there exists a locally nilpotent vector field δ on \tilde{X} for which $\delta_x = w$. Furthermore, the subgroup of automorphisms of \tilde{X} preserving $\pi^{-1}(z)$ acts transitively on $\pi^{-1}(z)$.*

Proof. Let $\varrho : X \rightarrow Q$ be a partial quotient associated with a nonzero locally nilpotent σ , x be general point in X and $q = \varrho(x)$ a general point in Q . In particular these points are smooth and one can choose local analytic coordinate systems at them. By [1, Theorem 4.14 and Remark 4.16] one can also choose an automorphism α which sends z to x such that in local analytic coordinate systems (v_0, \dots, v_{n-1}) (resp. (u_0, \dots, u_n)) at $q \in Q$ (resp. $x \in X$) one has $\alpha(Z)$ given by $u_{n-k} = \dots = u_n = 0$ and $\varrho^*(v_j) = u_j$ for $j \leq n - 1$. We replace Z by $\alpha(Z)$ and z by $\alpha(z)$ to make the argument local. In particular, $\pi^{-1}(z) \simeq \mathbb{P}^k$ has homogeneous coordinates $U_{n-k} : U_{n-k+1} : \dots : U_n$ such that $u_i U_j = u_j U_i$ for $n - k \leq i, j \leq n$. Without loss of generality consider the case when the vector w in $T_{\pi^{-1}(z)}$ is tangent to the line L with fixed relation $U_{n-k+1} : \dots : U_{n-1} : U_n$ where $U_n \neq 0$ and arbitrary U_{n-k} . Note that at the origin x_0 of the local coordinate system σ_{x_0} is proportional to the vector $\partial/\partial u_n$, i.e. we can suppose that $\sigma_{x_0} = \partial/\partial u_n$. Since $u_{n-k} = \varrho^*(v_{n-k}) \in \text{Ker } \sigma$ we see that $u_{n-k}\sigma$ is also locally nilpotent. Denote by

Φ the automorphism $\Phi = \exp(tu_{n-k}\sigma)$ of X for some value of parameter $t \in \mathbb{C}$. By [1, Lemma 4.1] we have

$$(5) \quad d_{x_0}\Phi(\nu) = \nu + tdu_{n-k}(\nu)\partial/\partial u_n$$

for every $\nu \in T_{x_0}X$. Since $u_{n-k}\sigma$ vanishes on Z it can be lifted as a locally nilpotent derivation δ on \tilde{X} . Furthermore, Formula (5) shows that the elements of the flow of δ preserve $\pi^{-1}(z) \simeq \mathbb{P}^k$ and act on as elementary transformations of form $(U_{n-k} : U_{n-k+1} : \dots : U_n) \rightarrow ((U_{n-k} + tU_n) : U_{n-k+1} : \dots : U_n)$. That is, the action induced by δ is a translation along the affine line $\mathbb{C} \simeq L \setminus \{U_{n-k} = \infty\}$ which yields the first statement. The fact that elementary transformations generate a special linear group implies the second statement and we are done. \square

4. AFFINE MODIFICATIONS

The next definition of affine modifications and their properties can be found in [13].

Definition 4.1. A birational morphism $\varphi : X' \rightarrow X$ of affine algebraic varieties is called an affine modification. In particular, one can find effective reduced divisors $D \subset X$ and $E \subset X'$ such that the restriction $\varphi|_{X' \setminus E} : X' \setminus E \rightarrow X \setminus D$ is an isomorphism. Though these divisors are not determined uniquely it will be clear from the context below what they are. We call E the exceptional divisor of the modification and D the divisor of the modification. Furthermore, we consider only the cases when D is principal, i.e. $D = f^*(0)$ for some regular function f from the ring $A = \mathbb{C}[X]$ of regular functions on X . In this case the ring $A' = \mathbb{C}[X']$ of regular functions on X' is generated over A by functions of form g/f where g runs over an ideal I of A , i.e. $A' = A[I/f]$. Even for a fixed f this ideal may not be unique and to remove this ambiguity we suppose that it is the largest ideal for which $A' = A[I/f]$. This largest ideal will be called the ideal of the modification. The center of modification (for fixed D and E) is the closure Z of $\varphi(E)$ in X .

Remark 4.2. The geometrical meaning of the modification is the following. One consider the blowing up $\pi : \tilde{X} \rightarrow X$ of X along the ideal I and obtain X' by removing from \tilde{X} those divisors on which the zero multiplicity of f is greater than the zero multiplicity of at least one function g from I (and letting $\varphi = \pi|_{X'}$). In particular, if D and Z are smooth and the ideal of the modification coincides with the defining ideal $I(Z)$ of the center then X' is the complement in \tilde{X} to the proper transform of D .

Lemma 4.3. Let $\varrho : X \rightarrow Q$ be a dominant morphism of normal irreducible affine algebraic varieties, D be an effective principal divisor in Q , $Q_0 = Q \setminus D$, and $X_0 = \varrho^{-1}(Q_0)$. Suppose that X_0 is isomorphic to $Q_0 \times \mathbb{C}^m$ over Q_0 . Then

(a) there exists an affine modification $\varphi : X \rightarrow Q \times \mathbb{C}^m$ whose restriction over Q_0 is an isomorphism;

(b) for every locally nilpotent vector field σ on X_0 tangent to the fibers of $\varrho|_{X_0}$ there exists an equivalent³ locally nilpotent vector field δ that extends regularly to X ;

³We call two locally nilpotent vector fields equivalent if as derivations they have the same kernel.

(c) furthermore, for every $q \in Q_0$ the restrictions of the fields σ and δ to the fiber $\varrho^{-1}(q)$ differ by a nonzero constant factor.

Proof. Note that the isomorphism $X_0 \simeq Q_0 \times \mathbb{C}^m$ has a coordinate form $(\varrho, h_1, \dots, h_m)$ where each h_i is a regular function on X_0 . If these functions extend regularly to X then it suffices to put $\varphi = (\varrho, h_1, \dots, h_m)$. Otherwise, consider a regular function $g \in \mathbb{C}[Q]$ for which $D = g^*(0)$. Note that the extension of h_i to X can have poles only on the divisor $\varrho^{-1}(D)$. Thus for sufficiently large k_i the function $g^{k_i} h_i$ is regular on X (because of normality). Replacing every h_i by $g^{k_i} h_i$ we see that the same φ yields the desired affine modification in (a).

Similarly, since by the assumption $g \in \text{Ker } \sigma$ the field $\delta = g^k \sigma$ is also locally nilpotent and for k large enough it extends regularly to X . Thus we have (b). For (c) it suffices to observe that g does not vanish on Q_0 . □

Proposition 4.4. *Let the assumption of Lemma 4.3 hold, $D = g^{-1}(0)$, and $\tau : Q \times \mathbb{C}_u^m \rightarrow Q \times \mathbb{C}_u^m$ (where $\bar{u} = (u_1, \dots, u_m)$ is a coordinate system on \mathbb{C}^m) be a birational morphism over Q such that $\tau^*(\bar{u}) = \bar{u} + g^k \bar{e}$ where \bar{e} is a regular function on Q with values in \mathbb{C}^m . Then for k large enough this endomorphism can be lifted to a birational morphism $\theta : X \rightarrow X$, i.e. the following commutative diagram holds*

$$\begin{array}{ccc} X & \xrightarrow{\theta} & X \\ \downarrow \varphi & & \downarrow \varphi \\ Q \times \mathbb{C}^m & \xrightarrow{\tau} & Q \times \mathbb{C}^m. \end{array}$$

Furthermore, θ maps $\varphi^{-1}(D)$ isomorphically on itself.

Proof. By Lemma 4.3 one can consider an affine modification $\varphi : X \rightarrow Q \times \mathbb{C}^m$, i.e. $\mathbb{C}[X]$ is generated over $\mathbb{C}[Q \times \mathbb{C}^m]$ by elements of form I/g^k where k is some natural number and I is an ideal in $\mathbb{C}[Q \times \mathbb{C}^m]$ generated by g^{k_0} and elements $g_1, \dots, g_k \in \mathbb{C}[Q \times \mathbb{C}^m]$. By the assumption τ^* transfer g to g and I into the ideal J generated by g^{k_0} and $\tau^*(g_1), \dots, \tau(g_k)$. Choose $k \geq k_0$. By Taylor expansion one has $\tau(g_i) = g_i + g^k h_i$ where h_i is regular on $Q \times \mathbb{C}^m$. Hence, J is contained in I . Now the first statement follows from [13, Proposition 2.1].

Note also that τ is invertible in an étale neighborhood of D in $Q \times \mathbb{C}^m$. Hence θ is invertible in an étale neighborhood of $\varphi^{-1}(D)$ in X which yields the second statement. □

Corollary 4.5. *Let Q and X be algebraic varieties, $X_0 = Q \times \mathbb{C}_u^m$, $\varphi : X \rightarrow X_0$ be an affine modification over Q with divisor $D \subset X_0$ given by the zeros of a regular function $g \in \mathbb{C}[Q] \subset \mathbb{C}[X_0]$, and $f \in \mathbb{C}[Q] \subset \mathbb{C}[X_0]$ be another regular function that has disjoint zeros with those of g . Furthermore, suppose that $f|_D \equiv 1$ with multiplicity k . Let $\tau : X_0 \rightarrow X_0$ be the birational morphism over Q for which $\tau^*(\bar{u}) = f(q)\bar{u}$. Then for k large enough there exists a modification $\theta : X \rightarrow X$ over Q such that the commutative diagram and the second statement from Proposition 4.4 hold.*

Now we have the following observation.

Theorem 4.6. *Let the assumptions of Corollary 4.5 hold, $Z = \varphi^{-1}(Z_0)$ where Z_0 is given in X_0 by the equations $\bar{u} = \bar{0}$ and $f = 0$. Suppose that $\pi : \tilde{X} \rightarrow X$ is the blowing of X up along the center Z . Then \tilde{X} contains a Zariski open set X' isomorphic to X such that $X' \cap E$ is dense in the exceptional divisor $E = \pi^{-1}(Z)$ of π .*

Proof. Consider the affine modification $\psi : X' \rightarrow X$ along the divisor $f^{-1}(0)$ with center Z . That is, X' is a Zariski open subset of \tilde{X} with $X' \cap E$ dense in E . On the other hand $\theta : X \rightarrow X$ is also an affine modification with center Z . Its exceptional divisor contains $f^{-1}(0)$ and by Corollary 4.5 it contains nothing else. This yields the desired isomorphism $X \simeq X'$. □

Notation 4.7. Let X be a quasi-affine algebraic manifold, Z be a submanifold of X which is a strict complete intersection. That is, the defining ideal I of Z is generated by regular functions $f := g_0, g_1, \dots, g_k$ where $k+1$ is the codimension of Z in X . For $l < k$ consider the strict complete intersection $Z_1 \subset X$ given by $f = g_1 = \dots = g_l = 0$. Suppose that $\varphi : X'_1 \rightarrow X$ is the modification with center Z_1 and divisor $D = f^*(0)$ while $\pi_1 : \tilde{X}_1 \rightarrow X$ (resp. $\pi : \tilde{X} \rightarrow X$) is the blowing up of X with center at Z_1 (resp. Z), i.e. X'_1 can be viewed as a Zariski open subset of \tilde{X}_1 . Suppose that $\tilde{Z}_1 \subset X'_1$ is the complete strict intersection given by $f \circ \varphi = g_{l+1} \circ \varphi = \dots = g_k \circ \varphi = 0$ and $\pi' : \tilde{X}'_1 \rightarrow X'$ is the blowing up of X'_1 with center at \tilde{Z}_1 .

Proposition 4.8. *There is a natural birational morphism $\theta : \tilde{X}'_1 \rightarrow \tilde{X}$ such that $\theta(\tilde{X}'_1)$ meets the exceptional divisor E of π along a Zariski dense open subset of E .*

Proof. Recall that \tilde{X} can be viewed as the submanifold of $X \times \mathbb{P}^k$ given by equations $U_i g_j = U_j g_i$ for $i, j = 0, \dots, k$ where $(U_0 : U_1 : \dots : U_k)$ is a homogeneous coordinate system on \mathbb{P}^k . Similarly \tilde{X}_1 is the submanifold of $X \times \mathbb{P}^l$ given by equations $V_i g_j = V_j g_i$ for $i, j = 0, \dots, l$ where $(V_0 : V_1 : \dots : V_l)$ is a homogeneous coordinate system on \mathbb{P}^l . Note that X'_1 is given in \tilde{X}_1 by the equation $V_0 = 1$ and \tilde{X}'_1 is the submanifold of $X'_1 \times \mathbb{P}^{k-l}$ given by equations $W_i g_j = W_j g_i$ for $i, j = 0, l+1, l+2, \dots, k$ where $(W_0 : W_{l+1} : \dots : W_k)$ is a homogeneous coordinate system on \mathbb{P}^{k-l} .

Note that one has the natural birational map $\theta : \tilde{X}'_1 \dashrightarrow \tilde{X}$ over X which is an isomorphism over $X \setminus Z_1$ and regular over $X \setminus Z$. Thus one needs to check only the regularity over Z . However, one can see that over Z this map is automatically given by the following

$$[(V_0 : V_1 : \dots : V_l), (W_0 : W_{l+1} : \dots : W_k)] \rightarrow (U_0 : U_1 : \dots : U_k) = (V_0 W_0 : V_1 W_0 : \dots : V_l W_0 : W_{l+1} V_0 : \dots : W_k V_0)$$

which is regular since $V_0 = 1$. Note also that when $W_0 = 1$ then this morphism is an embedding. This yields the desired conclusion about the density of the intersection of E and $\theta(\tilde{X}'_1)$. □

5. k -FLEXIBILITY

Definition 5.1. A flexible quasi-affine manifold X will be called k -flexible for $k > 0$ if there exists a morphism $\varrho : X \rightarrow Q$ into a normal affine algebraic variety Q such that over a Zariski open dense subset Q_0 of Q the variety $\varrho^{-1}(Q_0)$ is isomorphic to $Q_0 \times \mathbb{C}^k$ over Q_0 .

Remark 5.2. (i) Note that for $l > k$ each l -flexible variety is automatically k -flexible.

(ii) Every flexible manifold X is, of course, 1-flexible, since one can consider any partial quotient morphism of $\varrho : X \rightarrow Q$. Then for some Q_0 as above $\varrho^{-1}(Q_0) \rightarrow Q_0$ is a locally trivial \mathbb{C} -fibration. Requiring that Q_0 is affine one can guarantee that it is in fact an line bundle. Hence removing from Q_0 a divisor we make $\varrho^{-1}(Q_0)$ the desired direct product.

(iii) For $k = 2$ it is also enough to require that $\varrho : X \rightarrow Q$ has general fibers isomorphic to \mathbb{C}^2 . The existence of a desired Q_0 follows from [14].

Example 5.3. Consider a hypersurface H given by $uv = p(\bar{x})$ in \mathbb{C}^{n+2} where u, v , and $\bar{x} = (x_1, \dots, x_n)$ are coordinates on \mathbb{C}^{n+2} . If the zero locus of p in $\mathbb{C}_{\bar{x}}^n$ is smooth then H is a flexible manifold [13]. Note that $\dim H = n + 1$ and H is n -flexible.

Proposition 5.4. Let $X = SL(n)$, i.e. $\dim X = n^2 - 1$. Then X is k -flexible where $k = n^2 - n$.

Proof. Let $A = [a_{ij}]_{i,j=1}^n$ be a matrix from $SL(n)$ and let $\{A_{ij}\}$ be cofactors of this matrix. Consider the natural morphism of $\pi : X \rightarrow H$ into the hypersurface H given by the equation

$$(6) \quad a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n} = 1$$

in \mathbb{C}^{2n} with coordinates $(a_{11}, a_{12}, \dots, a_{1n}, A_{11}, A_{12}, \dots, A_{1n})$. Note that $\dim H = 2n - 1$. Let A' (resp. A'') be the matrix obtained from A by removing the first row (resp. the first row and the first column). Consider the action of $SL(n - 1)$ on X such that for $B \in SL(n - 1)$ the matrix BA is obtained by replacing A' by BA' while keeping the first row in A intact. This action is free and it preserves the fibers of π which are therefore of dimension at least $(n - 1)^2 - 1$. Observing the equality $n^2 - 1 = (n - 1)^2 - 1 + (2n - 1)$ one can see now that the fibers of π are nothing but the orbits of this action.

Furthermore, let H_0 be the complement to the zero locus of A_{11} in H . Since for every point A in $X' := \pi^{-1}(H_0)$ the determinant of A'' is nonzero, dividing the first row of A'' by this determinant we see that X' is isomorphic to $H_0 \times SL(n - 1)$. Note also that by Formula (6) $H_0 \simeq \mathbb{C}^* \times \mathbb{C}^{2n-2}$ where $\mathbb{C}^* \subset \mathbb{C}$ is equipped with the coordinate A_{11} . Hence $X' \simeq \mathbb{C}^* \times SL(n - 1) \times \mathbb{C}^{2n-2}$.

One can check that $SL(2)$ is 2-flexible. Thus by the induction assumption we can suppose that there are a morphism $\varrho_{n-1} : SL(n - 1) \rightarrow Q_{n-1}$ into a normal affine algebraic variety Q_{n-1} and a Zariski dense open subset Q_{n-1}^0 in it for which the Zariski open subset $\varrho_{n-1}^{-1}(Q_{n-1}^0)$ of $SL(n - 1)$ is isomorphic to $Q_{n-1}^0 \times \mathbb{C}^l$ with $l = (n - 1)^2 - n + 1$. This yields a Zariski open subset X_0 of X' isomorphic to $Q_n^0 \times \mathbb{C}^k$ where $k = l + 2n - 2 = n^2 - n$ and $Q_n^0 := \mathbb{C}^* \times Q_{n-1}^0$. Hence it suffices to modify this isomorphism $\psi : X_0 \simeq$

$Q_n^0 \times \mathbb{C}^k$ so that it becomes a restriction of an affine modification $X \rightarrow Q_n \times \mathbb{C}^k$ over some variety Q_n containing Q_n^0 as a Zariski open subset.

Consider morphism $\tau = (A_{11}, \varrho_{n-1}) : X' \rightarrow P := \mathbb{C} \times Q_{n-1}$. Without loss of generality we can suppose that $P \setminus Q_n^0$ is the principal divisor in P . Then modifying ψ we can extend it to a morphism $\varphi : X' \rightarrow P \times \mathbb{C}^k$ over P by Lemma 4.3(a).

Treat now Q_{n-1} as a closed subvariety of \mathbb{C}^m , i.e. P is a closed subvariety of \mathbb{C}^{m+1} with coordinates (u_0, u_1, \dots, u_k) where the lift of u_0 to X' coincides with A_{11} . Consider composition of τ with endomorphism $\theta : \mathbb{C}^{m+1} \rightarrow \mathbb{C}^{m+1}$ given by

$$(u_0, u_1, \dots, u_k) \rightarrow (u_0, u_0^N u_1, \dots, u_0^N u_k)$$

where N is natural. Since the restriction of θ to the complement to $u_0 = 0$ is an isomorphism we see that $\tau(Q_n^0)$ is isomorphic to its image $\theta \circ \tau(Q_n^0)$ which is, therefore, isomorphic to Q_n^0 . Furthermore, since $X \setminus X'$ is the zero locus of function A_{11} on X , for sufficiently large N this morphism $\theta \circ \tau$ extends to a morphism $\varrho_n : X \rightarrow Q_n$ where Q_n is the closure of $\theta \circ \tau(Q_n^0)$.

Applying Lemma 4.3(a) again we obtain the desired change of isomorphism $\psi : X_0 \simeq Q_n^0 \times \mathbb{C}^k$ so that it becomes an affine modification $X \rightarrow Q_n \times \mathbb{C}^k$ over Q_n which concludes induction and the proof. \square

Theorem 5.5. *Let X be a k -flexible quasi-affine manifold and let $\varrho : X \rightarrow Q$ and Q_0 be as in Definition 5.1. Suppose that $D_0 = Q \setminus Q_0$ is a principal divisor, Z is a closed submanifold of X of codimension $k+1$ such that $\varrho|_Z : Z \setminus \varrho^{-1}(D_0) \rightarrow Z_0$ is an isomorphism for $Z_0 = \overline{\varrho(Z)} \setminus D_0$ which is a principal divisor in Q_0 . Let $\pi : \tilde{X} \rightarrow X$ be the blowing of X up along the center Z .*

Then

- (i) \tilde{X} contains a Zariski open set X'_0 isomorphic to $X_0 = Q_0 \times \mathbb{C}^k$ such that $X'_0 \cap E$ is dense in the exceptional divisor $E = \pi^{-1}(Z)$ of π .
- (ii) Furthermore, if Z_0 is closed in Q then \tilde{X} contains a Zariski open set X' isomorphic to X (i.e. X' is k -flexible) and with $X' \cap E$ being dense in E .

Proof. The first statement follows from Theorem 4.6 in the case of $X = X_0$. The second statement also follows from Theorem 4.6 since by the assumption the Serre Theorem A implies that one can choose a regular function $f \in \mathbb{C}[Q]$ that vanishes on Z_0 and equal to 1 on D_0 with any prescribed multiplicity. \square

Remark 5.6. (1) Consider statement (ii) of Theorem 5.5. Because of flexibility for any point $x \in X' \cap E$ and every nonzero vector $v \in T_x X'$ there exists a locally nilpotent vector σ field on X' for which $\sigma|_x = v$ [1, Corollary 4.3]. In particular, this field can be chosen transversal to E .

(2) Suppose that $Q_0 = Q_1 \times \mathbb{C}_u$ in the case of Theorem 5.5 (i) and that the natural projection $Z_0 \rightarrow Q_1$ is étale. Then the lift of the vector field $\partial/\partial u$ to X'_0 is also locally nilpotent and transversal to $X'_0 \cap E$ at every point.

By the virtue of [1, Proposition 4.14 and Remark 4.16] we have the following.

Proposition 5.7. *Let X be a k -flexible manifold and let $\varrho : X \rightarrow Q$ and Q_0 be as in Definition 5.1. Suppose that $x \in X$ is such that $\varrho(x) \in Q_0$, $F = \varrho^{-1}(\varrho(x))$, and V is a k -dimensional subspace of $T_x X$. Then there exists an automorphism $\alpha \in \text{Aut}(X)$ such that $\alpha(x) = x$, $\alpha_*(T_x F) = V$, and furthermore α_* transforms a given basis of $T_x F$ into a given basis of V .*

6. MAIN THEOREMS

Now we are prepared for our main results.

Theorem 6.1. *Let X be a locally k -flexible manifold for $k \geq 2$, and Z be a closed submanifold of X of codimension at most k . Suppose that $\pi : \tilde{X} \rightarrow X$ is the blowing up of X along Z . Then \tilde{X} is subelliptic.*

Proof. For the proof we can assume X is k -flexible and by Remark 5.2(i) we can suppose that $\text{codim}_X Z = k$. Choose any point $z \in Z$ and any point $w \in \pi^{-1}(z)$. We need to construct a family of sprays on \tilde{X} of rank 1 that is dominating at w . By Proposition 3.4 the phase flow of a complete vector field on \tilde{X} can move w in a general position $\pi^{-1}(z)$, i.e. we can suppose that w is general. Consider $\varrho : X \rightarrow Q$ and Q_0 as in Definition 5.1, i.e. $X_0 := \varrho^{-1}(Q_0)$ is isomorphic to $Q_0 \times \mathbb{C}^k$. Choose a morphism $\tau : X \rightarrow Q \times \mathbb{C}_u$ such that $\tau|_{X_0} = (\varrho, \lambda)$ where $\lambda : \mathbb{C}^k \rightarrow \mathbb{C}$ is a linear map.

By Theorem 2.4, applying an automorphism we can suppose that $\varrho(z) \in Q_0$, $\tau|_Z : Z \rightarrow \tau(Z) =: Y$ is birational, Y is a hypersurface smooth at $y = \tau(z)$, and the projection $Y \rightarrow Q$ is smooth at y (in particular, the vector field $\partial/\partial u$ is transversal to Y at y). By Lemma 4.3 replacing this field $\partial/\partial u$ by an equivalent one δ we can extend it to X . Let $X_0 \simeq Q_0 \times \mathbb{C}^k$ and X'_0 be as in Theorem 5.5. Without loss of generality we can suppose that $Y \cap (Q_0 \times \mathbb{C})$ is smooth and is given in $Q_0 \times \mathbb{C}$ by the zero locus of a regular function f on $Q_0 \times \mathbb{C}$. Then $f \circ \pi$ yields a regular function on X'_0 whose zero locus may be viewed as a Zariski open subset W of $\pi^{-1}(Z)$ (and, moreover, this locus contains a Zariski open subset of $\pi^{-1}(z)$ for every $z \in Z$). Since w is general we have $w \in W$. Observe that because $X'_0 \simeq X_0$ the field δ has a lift to a locally nilpotent vector field σ on X'_0 which is transversal to $\pi^{-1}(Z)$ at w . By Proposition 1.4 σ extends to a spray of rank 1 on \tilde{X} and the only thing we have to show that the vector σ_w can be chosen general.

This follows from Proposition 5.7 because we can transform the \mathbb{C}^k -fibration ϱ by some automorphism α into another \mathbb{C}^k -fibration such that $\alpha_*(\sigma_z)$ is a general vector. This yields the desired conclusion. \square

As a corollary, we now give the proof of Theorem 0.3 in the introduction.

Proof. For the proof, we can assume that X is stably flexible, i.e. X is quasi-affine and $Y := X \times \mathbb{C}^N$ is flexible for some positive integer N . Then, Y is N -flexible, here we can choose $Q_0 = X$. Since the product of two flexible manifolds is again a flexible manifold, we can assume that $N \geq \dim(X)$. Consider $Z_1 = Z \times \mathbb{C}^N$, then Z_1 is a smooth algebraic submanifold of Y , being of codimension $\leq \dim(X) \leq N$.

Theorem 6.1 implies that the blowup $\pi_1 : \tilde{Y} \rightarrow Y$ at Z_1 is algebraically subelliptic. Since $\tilde{Y} = \tilde{X} \times \mathbb{C}^N$, it follows from the descent property for algebraic subellipticity that \tilde{X} is itself algebraically subelliptic, as desired. \square

Here are some final remarks.

Remark 6.2. (1) Note that $X = \mathbf{SL}_n$ is not contained in class \mathcal{A}_0 (or \mathcal{A}), i.e. it cannot be covered by open sets isomorphic to \mathbb{C}^N (where $N = \dim X$). Indeed, \mathbf{SL}_n is factorial since the ring of regular function on every simply connected algebraic group is a factorial domain (e.g, see [17]). Thus, if one assumes existence of an open subset $U \simeq \mathbb{C}^N$ such that $U \neq X$ then $D = X \setminus U$ must be a divisor because of affineness. Factoriality implies that $D = f^{-1}(0)$ for a regular function on X . However this function must be constant on U because of the fundamental theorem of algebra and, thus, on X . A contradiction.

(2) Let $H \subset \mathbb{C}_{u,v,\bar{x}}^{n+2}$ be a hypersurface given by $uv = p(\bar{x})$ in the case when the zero locus of p is smooth connected. Then it is again factorial because of the Nagata lemma (e.g., [2]). Thus if H is not isomorphic to \mathbb{C}^{n+1} it does not belong to class \mathcal{A} by the same argument as before.

REFERENCES

- [1] I. V. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, M. Zaidenberg: *Flexible varieties and automorphism groups*. Duke Math. J. 162:4 (2013), 767–823.
- [2] D. Eisenbud: *Commutative algebra. With a view toward algebraic geometry*. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995. xvi+785 pp.
- [3] P. Feller, I. van S. Stampfli, *Uniqueness of Embeddings of the Affine Line into Algebraic Groups*, preprint (2016) math.AG.arXiv:1609.02113.
- [4] H. Flenner, S. Kaliman, and M. Zaidenberg, *On Gromov-Winkelmann type theorem for flexible varieties*, J. of European Math. Soc. (to appear).
- [5] F. Forstnerič, *Stein Manifolds and Holomorphic Mappings. The Homotopy Principle in Complex Analysis*, Springer-Verlag, Berlin-Heidelberg, 2011.
- [6] F. Forstnerič, *The Oka principle of sections of subelliptic submersions* Math. Z. 241, 527551 (2002)
- [7] G. Freudenberg, *Algebraic Theory of Locally Nilpotent Derivations* (Encyclopaedia of Mathematical Sciences) Springer Berlin-Heidelberg-New York, 2006.
- [8] M. H. Gizatullin, *Quasihomogeneous affine surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 1047–1071.
- [9] M. Gromov: *Oka’s principle for holomorphic sections of elliptic bundles*. J. Amer. Math. Soc. 2 (1989), 851–897.
- [10] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York-Heidelberg, 1977.
- [11] S. Kaliman, *Extensions of isomorphisms between affine algebraic subvarieties of k^n to automorphisms of k^n* , Proc. Amer. Math. Soc. 113 (1991), no. 2, 325–334.
- [12] S. Kaliman, *Actions of \mathbb{C}^* and \mathbb{C}_+ on affine algebraic varieties*, Algebraic geometry Seattle 2005. Part 2, 629–654, Proc. Sympos. Pure Math. 80, Part 2, Amer. Math. Soc., Providence, RI, 2009.
- [13] S. Kaliman, M. Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism group*, Transform. Groups 4 (1999), 53–95.
- [14] S. Kaliman, M. Zaidenberg, *Families of affine planes: the existence of a cylinder*, Michigan Math. J. 49 (2001), no. 2, 353–367.

- [15] T. Kambayashi, D. Wright, *Flat families of affine lines are affine-line bundles*, Illinois J. Math. 29 (1985), 672–681.
- [16] F. Lárusson, T. T. Truong, *Algebraic subellipticity and dominability of blow-ups of affine spaces*, accepted in Documenta Mathematica. Preprint, 10p. (2016), arXiv:1606.08115.
- [17] V. L. Popov, *Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles*, Math USSR Izvestija 8 (1974), 301–327.
- [18] V. L. Popov, E. B. Vinberg: *Invariant Theory*. In: Algebraic geometry IV, A. N. Parshin, I. R. Shafarevich (eds.), Berlin, Heidelberg, New York: Springer-Verlag, 1994.
- [19] C. P. Ramanujam, *A note on automorphism groups of algebraic varieties*, Math. Ann. 156 (1964), 25–33.

UNIVERSITY OF MIAMI, CORAL GABLES, FL 33124, USA
E-mail address: `kaliman@math.miami.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BERN, BERN, SWITZERLAND
E-mail address: `frank.kutzschebauch@math.unibe.ch`

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF ADELAIDE, ADELAIDE SA 5005, AUSTRALIA
E-mail address: `tuyen.truong@adelaide.edu.au`